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Absolute minima of the Higgs potential for the 75 of SU(5)

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Abstract. The problem of minimising the Higgs potential for the 75-dimensional representation of SU(5) is considered. The most general form of the potential is established. In the absence of a cubic term it is shown that the possible symmetry breaking patterns include SU(3) × SU(2) × U(1), and a new counterexample to Michel's conjecture is found.

1. Introduction

It is widely believed that a gauged field theory with broken symmetry describes the fundamental interactions of quarks and leptons. In models such as the standard SU(3) × SU(2) × U(1) or minimal SU(5) the symmetry breaking is produced via the Higgs mechanism. Quite generally the scalar Higgs field, φ, transforms as some representations ν_G (possibly reducible) of the gauge group G. The corresponding Higgs potential $V(\varphi)$ is a fourth degree G invariant polynomial in the components of φ, which is required to be bounded from below and, in the broken symmetry phase, to have a maximum at the origin. In these circumstances the symmetry is broken by the Higgs field φ acquiring a non-zero vacuum expectation value $\langle\varphi_0\rangle$ equal to the value of φ which minimises $V(\varphi)$. The symmetry is broken from G to the subgroup G_{φ_0} of G which leaves $\langle\varphi_0\rangle$ fixed. G_{φ_0} is called the little group of $\langle\varphi_0\rangle$, and we will call the Lie algebra of G_{φ_0} the little algebra. The task of classifying all the possible symmetry breaking patterns in the case of Higgs fields transforming as an irreducible representation ν_G of a simple gauge group G was initiated by Li (1974) and extended by many others (Rüegg 1980, Kim 1982, Jetzer *et al* 1984, Cummins and King 1984).

General approaches to the problem have been used to provide important theorems (Michel and Radicati 1971, Michel 1979, Abud and Sartori 1983) and a conjecture due to Michel (1979) would have considerably simplified the problem. Recently, however, counterexamples to this conjecture have been found amongst finite groups by Jaric (1983) and amongst continuous Lie groups by Abud *et al* (1984b). This latter counterexample involves the **75** of SU(5).

A role for this representation has also been found within an SU(5) model in a study of fermion masses by Barbieri *et al* (1981a, b), (and an alternative SU(5) model based on the **75** has been proposed by Hubsch and Pallua (1984)). The group theoretical merits of the **75** have been pointed out by Tsao (1981).

Unlike the situation for the **24** on restriction of the **75** from SU(5) to SU(3) × SU(2) × U(1) the only singlet of colour SU(3) is also an SU(2) singlet. This militates against any deviation from the mass formula $M_w = M_z \cos\theta_w$ associated with the subsequent breaking to SU(3) × U(1). In addition any tendency to break from SU(5) to SU(4) ×

U(1) is suppressed. The first of these points favouring the **75** over the **24** is also made by Georgi (1982) in an SU(5) supersymmetric context.

Thus there are two motivations, one physical and one mathematical, for studying the minimisation of the SU(5) invariant potential $V(\varphi)$ associated with the representation **75**. Rather than tackle the most general such potential of degree four it is convenient to impose an additional Z_2 invariance corresponding to the discrete transformation $\varphi \to -\varphi$ in order to eliminate the cubic term from the potential (Abud *et al* 1984a, Hubsch *et al* 1984). When all the relationships between fourth-order invariants are used it is found that the minimisation reduces to a two-dimensional problem of the type described by Kim (1982).

The form of the potential is given in § 2, while the justification for this form may be found in the two appendices. Following the methods of Kim (1982) the minimisation of the potential is undertaken in § 3, where it is shown that the absolute minima may be found from a knowledge of the boundary of a closed, bounded, connected subset Ω of $\mathbb{R}^2$. The nature of Ω and its boundary, $\partial\Omega$, is discussed in more detail in § 4, leading to a description of possible symmetry breaking patterns including SU(3) × SU(2) × U(1). For the most part we adhere to the notation of Abud *et al* (1984b), and we thank the authors for informing us of their work prior to publication.

2. The Higgs potential

The 75-dimensional representation of SU(5) may be realised by the action of SU(5) on the traceless mixed tensor φ^{ab}_{cd} satisfying the constraints

$$\varphi^{ab}_{cd} = -\varphi^{ba}_{cd} = -\varphi^{ab}_{dc} = \varphi^{ba}_{dc} \tag{2.1a}$$

$$\varphi^{ab}_{ac} = 0 \tag{2.1b}$$

$$\varphi^{ab}_{cd} = (\varphi^{cd}_{ab})^* \tag{2.1c}$$

where the indices takes the values $1, \ldots, 5$. Note that the representation is real. If we ignore for the moment any non-trivial relationships that may exist between invariants then it is not difficult to see that the most general form of the potential is (Abud *et al* 1984b)

$$V(\varphi) = -\mu^2 Q + \nu_1 C + \nu_2 C' + \sum_{i=1}^{6} \lambda_i K_i \tag{2.2}$$

where

$$\begin{aligned} &Q = B^a_a \qquad && C = A^{ab}_{cd}\varphi^{cd}_{ab} \qquad && C' = D^{ab}_{cd}\varphi^{cd}_{ab} \\ &K_1 = Q^2 \qquad && K_2 = A^{ab}_{cd}A^{cd}_{ab} \qquad && K_3 = B^a_b B^b_a \\ &K_4 = -A^{ab}_{cd}D^{cd}_{ab} \qquad && K_5 = -D^{ab}_{cd}D^{cd}_{ab} \qquad && K_6 = -D^{ab}_{cd}D^{dc}_{ab} \end{aligned} \tag{2.3}$$

and where

$$A^{ab}_{cd} = \varphi^{ab}_{ef}\varphi^{ef}_{cd} \qquad D^{ab}_{cd} = \varphi^{ae}_{cf}\varphi^{bf}_{de} \qquad B^a_b = A^{ac}_{bc}. \tag{2.4}$$

However in the case of SU(5) it is shown in appendix 1 that the number of algebraically independent invariants of degrees 2, 3 and 4 are respectively 1, 1 and 3. Thus not all the terms in (2.2) are independent. In appendix 2 the following identities are derived:

$$C=4C' \qquad K_2-2K_3+8K_4+4K_6=0 \tag{2.5a,b}$$

$$K_2-K_3+5K_4+2K_5=0 \qquad Q^2-8K_3+2K_2+8K_4=0. \tag{2.5c,d}$$

These are in agreement with the relationships of Abud *et al* (1984b), but include an additional fourth-order equation, which considerably simplifies the problem. The relationships (2.5) may be used to eliminate C', K_4, K_5 and K_6 from (2.2).

The potential may be further simplified by imposing the discrete symmetry condition

$$V(-\varphi)=V(\varphi) \tag{2.6}$$

which eliminates the cubic terms at the expense of enlarging the symmetry group to $SU(5)\times Z_2$.

Having done this the most general Higgs potential takes the simple form

$$V(\varphi)=-\mu^2Q+\lambda_1K_1+\lambda_2K_2+\lambda_3K_3 \tag{2.7}$$

with

$$K_1=Q^2=|\varphi^{ab}_{cd}\varphi^{cd}_{ab}|^2=|\varphi|^4. \tag{2.8}$$

Note though that the full linear symmetry group of (2.7) is $U(5)\times Z_2$.

3. Minimisation of the potential

The minimisation of potentials of the form (2.7) has been described as follows in a very convenient geometrical manner by Kim (1982).

Introducing the homogeneous variables

$$X=K_2/Q^2 \qquad Y=K_3/Q^2 \tag{3.1}$$

the potential (2.7) is given by

$$V(\varphi)=-\mu^2Q+Q^2(\lambda_1+\lambda_2X+\lambda_3Y). \tag{3.2}$$

As φ takes values in $\mathbb{R}^{75}$, $V(\varphi)$ depends only on the three parameters Q, X and Y, with X and Y constrained by (3.1) to lie in some region Ω of $\mathbb{R}^2$. Since a change in Q corresponds to an overall scaling of φ, X and Y are independent of Q. Thus we may regard (3.1) as a map $S^{74}\rightarrow\mathbb{R}^2$ with image Ω and it follows that Ω is a closed, bounded, connected subset of $\mathbb{R}^2$.

With $\mu^2>0$, $V(\varphi)$ has the required maximum at $\varphi=\mathbf{0}$. $V(\varphi)$ will be bounded from below as $|\varphi|\rightarrow\infty$ provided that

$$\lambda_1+\lambda_2X+\lambda_3Y>0. \tag{3.3}$$

This must hold for all φ as $|\varphi|\rightarrow\infty$ and hence for all X and Y in Ω, thus putting constraints on λ_1, λ_2 and λ_3. (Note though that (3.3) may always be satisfied by choosing λ_1 sufficiently large.)

Since for $Q>0$

$$\partial V/\partial Q=-\mu^2+2Q(\lambda_1+\lambda_2X+\lambda_3Y) \tag{3.4}$$

and

$$\partial^2V/\partial Q^2=2(\lambda_1+\lambda_2X+\lambda_3Y)>0 \tag{3.5}$$

minima of $V(\varphi)$ with respect to Q occur when

$$Q_{\min} = |\varphi|^2_{\min} = \mu^2/2(\lambda_1 + \lambda_2 X + \lambda_3 Y) \tag{3.6}$$

and the corresponding value of the potential is

$$V(\varphi) = -\tfrac{1}{4}\mu^4/(\lambda_1 + \lambda_2 X + \lambda_3 Y) = -\tfrac{1}{2}\mu^2 |\varphi|^2_{\min}. \tag{3.7}$$

Furthermore in the interior of Ω

$$\partial V/\partial X = \lambda_2 Q^2 \qquad \text{and} \qquad \partial V/\partial Y = \lambda_3 Q^2$$

so that it is clear that all the minima of $V(\varphi)$, including the absolute minima, occur on the boundary $\partial\Omega$ of Ω. The minimisation of $V(\varphi)$ thus reduces to a study of $\partial\Omega$. Furthermore it is clear from (3.7) that in fact the minimisation of $V(\varphi)$ is equivalent to the minimisation of the *linear* function $\lambda_1 + \lambda_2 X + \lambda_3 Y$ on Ω. This makes it particularly easy to find the ranges of parameters for which any particular solution is an absolute minimum.

It is perhaps worth noting that although it is easy to find absolute minima by the above method, it is not in general possible to identify all local minima.

4. The boundary of Ω

To investigate the region Ω, two constraints on X and Y are established. Consideration of the eigenvalues of an arbitrary $n \times n$ Hermitian matrix M shows that

$$\mathrm{Tr}(M^2)/(\mathrm{Tr}\, M)^2 \geqslant 1/n. \tag{4.1}$$

Equality holds if and only if

$$M^i{}_j = m\delta^i{}_j \qquad i, j = 1, \ldots, n. \tag{4.2}$$

However

$$X = \frac{K_2}{Q^2} = \frac{A^{ab}_{cd} A^{cd}_{ab}}{[A^{ab}_{ab}]^2} = \frac{\mathrm{Tr}(A^2)}{(\mathrm{Tr}\, A)^2} \tag{4.3}$$

and

$$Y = \frac{K_3}{Q^2} = \frac{B^a_b B^b_a}{[B^a_a]^2} = \frac{\mathrm{Tr}(B^2)}{(\mathrm{Tr}\, B)^2} \tag{4.4}$$

with A and B Hermitian 10×10 and 5×5 matrices respectively. Hence

$$X \geqslant \tfrac{1}{10} \tag{4.5a}$$

and

$$Y \geqslant \tfrac{1}{5} \tag{4.5b}$$

for all φ. Note that if $X = \frac{1}{10}$ then $A^{ab}_{cd} \propto \delta_{[}{}^a_c \delta^b_{d]}$, which implies from (2.3) that $B^a_b \propto \delta^a_b$. So that if the first bound (4.5a) is saturated so is the second (4.5b). It is possible to write down a vector φ_0 with this property which thus maps to a point on $\partial\Omega$, namely

$$\varphi_0 = \alpha_0 \{{}^{12}_{12} - {}^{13}_{13} + {}^{14}_{14} - {}^{15}_{15} - {}^{23}_{23} - {}^{24}_{24} + {}^{25}_{25} + {}^{34}_{34} + {}^{35}_{35} - {}^{45}_{45}\} \tag{4.6}$$

where α_0 is a non-zero constant. The notation is that of Abud *et al* (1984b), whereby the other non-zero components of φ_0 are generated using (2.1a). Thus $\varphi^{12}_{12} = -\varphi^{21}_{12} = -\varphi^{12}_{21} = \varphi^{21}_{21} = \alpha_0$ etc (see appendix 3 for more details of this notation).

It is in fact possible to saturate (4.5*b*) with (see table 3)

$$\varphi(t)=-{}^{14}_{14}+{}^{15}_{15}+{}^{23}_{23}-{}^{25}_{25}-{}^{35}_{35}+{}^{45}_{45}+\sqrt{2}[{}^{23}_{14}+{}^{14}_{23}]+t[{}^{12}_{34}+{}^{34}_{12}]+t[{}^{13}_{24}+{}^{24}_{13}] \qquad 0\leqslant t\leqslant 1. \tag{4.7}$$

This maps to the straight line $Y=0.2$, $0.1\leqslant X\leqslant 0.22$ and in view of (4.5*b*) this must be part of $\partial\Omega$.

The end point $Y=0.2$, $X=0.22$ corresponds to

$$\varphi_7=-{}^{14}_{14}+{}^{15}_{15}+{}^{23}_{23}-{}^{25}_{25}-{}^{35}_{35}+{}^{45}_{45}+\sqrt{2}[{}^{23}_{14}+{}^{14}_{23}] \tag{4.8}$$

and there appear to be no points on the line $Y=0.2$ with larger X values. Note that φ_4 (see tables 1 and 2) also satisfies $Y=0.2$; it does not however have the same little algebra as (4.7). It follows that, at least at one point, vectors in two distinct strata map to the same point on $\partial\Omega$.

It does not seem possible to find any more useful algebraic constraints on X and Y for the form (4.5). We thus turn to group theoretical and numerical methods. Previous studies of similar problems (Kim 1984) suggest that maximal little groups with one invariant are frequently associated with cusp points on $\partial\Omega$. The maximal little algebras of SU(5) together with their invariant vectors appear in tables 1 and 2 and figure 1 (see also Abud *et al* 1984b). Their images are shown in figure 2 and it does indeed appear, as will be shown, that these points lie on $\partial\Omega$. Plotting random points strongly suggests that the line from (0.1, 0.2) to $(\frac{1}{6},\frac{1}{4})$, i.e. from φ_0 to φ_3 is a boundary, and it is possible to find a curve, $\varphi(t)$, that maps to this line (see table 3)

$$\begin{aligned}\varphi(t)=&-{}^{12}_{12}+{}^{13}_{13}+t{}^{14}_{14}-t{}^{15}_{15}-t{}^{23}_{23}+{}^{24}_{24}+t{}^{25}_{25}-{}^{34}_{34}\\&+t{}^{35}_{35}-t{}^{45}_{45}+(1-t^2)^{1/2}[{}^{14}_{23}+{}^{23}_{14}] \qquad 0\leqslant t\leqslant 1.\end{aligned} \tag{4.9}$$

Table 1. Description of vectors appearing in figure 2.

Components of φ	$K_2=A^{ab}_{cd}A^{cd}_{ab}$	$K_3=B^a_bB^b_a$	$Q=\varphi^{ab}_{cd}\varphi^{cd}_{ab}$	$X=K_2/Q^2$	$Y=K_3/Q^2$
$\varphi_0={}^{12}_{12}+{}^{13}_{13}-{}^{14}_{14}-{}^{15}_{15}-{}^{23}_{23}+{}^{24}_{24}-{}^{25}_{25}-{}^{34}_{34}+{}^{35}_{35}+{}^{45}_{45}$	160	320	40	0.100	0.200
$\varphi_1={}^{24}_{24}+{}^{25}_{25}+{}^{45}_{45}-{}^{12}_{12}-{}^{14}_{14}-{}^{15}_{15}-{}^{23}_{23}-{}^{34}_{34}-{}^{35}_{35}+3\binom{13}{13}$	1 440	1 344	72	0.278	0.259
$\varphi_2=3\binom{14}{23}+3\binom{23}{14}+2\binom{14}{14}+2\binom{23}{23}-{}^{12}_{12}-{}^{13}_{13}-{}^{24}_{24}-{}^{34}_{34}$	10 080	3 600	120	0.700	0.250
$\varphi_3={}^{13}_{13}+{}^{24}_{24}-{}^{12}_{12}-{}^{34}_{34}+{}^{14}_{23}+{}^{23}_{14}$	96	144	24	0.167	0.250
$\varphi_4=3\binom{12}{12}+3\binom{13}{13}-\binom{14}{14}-5\binom{15}{15}-3\binom{23}{23}+\binom{24}{24}$ $-\binom{25}{25}-3\binom{34}{34}+3\binom{35}{35}+3\binom{45}{45}+4[{}^{15}_{24}+{}^{24}_{15}]$ $+2\sqrt{6}[{}^{14}_{23}+{}^{23}_{14}+{}^{25}_{34}+{}^{34}_{25}]$	124 320	141 120	840	0.176	0.200
$\varphi_5=-{}^{12}_{12}-{}^{14}_{14}-{}^{23}_{23}-{}^{34}_{34}+2[{}^{13}_{13}+{}^{24}_{24}]$	576	576	48	0.250	0.250
$\varphi_6={}^{14}_{14}+{}^{23}_{23}-4[{}^{12}_{12}+{}^{34}_{34}]+5[{}^{14}_{23}+{}^{23}_{14}]$ $+\sqrt{21}[{}^{15}_{15}+{}^{35}_{35}-{}^{25}_{25}-{}^{45}_{45}]$ $+[3-\sqrt{21}]{}^{13}_{13}+[3+\sqrt{21}]{}^{24}_{24}$	114 240	178 704	912	0.138	0.215
$\varphi_7=-{}^{14}_{14}+{}^{15}_{15}+{}^{23}_{23}-{}^{25}_{25}-{}^{35}_{35}+{}^{45}_{45}+\sqrt{2}[{}^{23}_{14}+{}^{14}_{23}]$	352	320	40	0.220	0.200
$\varphi_8={}^{14}_{23}+{}^{23}_{14}$	32	16	8	0.500	0.250

Note. φ_5, φ_6 and φ_8 do not lie on $\partial\Omega$. The first two play an important role in Abud *et al* (1984a, b) and are included for reference. φ_8 is included for completeness.

Table 2. Little algebras of vectors appearing in table 1.

φ	Little algebra	Generators
φ_0	U(1)+U(1)+U(1)+U(1)[a]	${}^{1}_{1}-{}^{2}_{2},\ {}^{2}_{2}-{}^{3}_{3},\ {}^{3}_{3}-{}^{4}_{4},\ {}^{4}_{4}-{}^{5}_{5}$
φ_1	SU(3)+SU(2)+U(1)[b]	SU(2): ${}^{1}_{1}-{}^{3}_{3},\ {}^{1}_{3}\pm{}^{3}_{1}$ SU(3): ${}^{2}_{2}-{}^{4}_{4},\ {}^{4}_{4}-{}^{5}_{5},\ {}^{2}_{4}\pm{}^{4}_{2},\ {}^{2}_{5}\pm{}^{5}_{2},\ {}^{4}_{5}\pm{}^{5}_{4}$ U(1): $2\binom{2}{2}+2\binom{4}{4}+2\binom{5}{5}-3\binom{1}{1}-3\binom{3}{3}$
φ_2	Sp(4)+U(1)[b]	Sp(4): ${}^{1}_{1}-{}^{2}_{2}+{}^{3}_{3}-{}^{4}_{4},\ {}^{1}_{1}+{}^{2}_{2}-{}^{3}_{3}-{}^{4}_{4},\ {}^{1}_{4}\pm{}^{4}_{1},\ {}^{2}_{3}\pm{}^{3}_{2}$ ${}^{2}_{1}\pm{}^{1}_{2}-{}^{4}_{3}\mp{}^{3}_{4},\ {}^{1}_{3}\pm{}^{3}_{1}+{}^{2}_{4}\pm{}^{4}_{2}$ U(1): ${}^{1}_{1}+{}^{2}_{2}+{}^{3}_{3}+{}^{4}_{4}-4\binom{5}{5}$
φ_3	SU(2)+SU(2)+U(1)[b]	SU(2): ${}^{1}_{2}\pm{}^{2}_{1}+{}^{3}_{4}\pm{}^{4}_{3},\ {}^{2}_{2}-{}^{1}_{1}+{}^{4}_{4}-{}^{3}_{3}$ SU(2): ${}^{3}_{1}\pm{}^{1}_{3}+{}^{4}_{2}\pm{}^{2}_{4},\ {}^{3}_{3}-{}^{1}_{1}+{}^{4}_{4}-{}^{2}_{2}$ U(1): ${}^{1}_{1}+{}^{2}_{2}+{}^{3}_{3}+{}^{4}_{4}-4\binom{5}{5}$
φ_4	SU(2)[b]	$\sqrt{2}[{}^{1}_{2}\pm{}^{2}_{1}+{}^{4}_{5}\pm{}^{5}_{4}]+\sqrt{3}[{}^{2}_{3}\pm{}^{3}_{2}+{}^{3}_{4}\pm{}^{4}_{3}]$ $4\binom{1}{1}+2\binom{2}{2}-2\binom{4}{4}-4\binom{5}{5}$
φ_5	SU(2)+SU(2)+U(1)+U(1)[a]	SU(2): ${}^{1}_{3}\pm{}^{3}_{1},\ {}^{1}_{1}-{}^{3}_{3}$ SU(2): ${}^{2}_{4}\pm{}^{4}_{2},\ {}^{4}_{4}-{}^{2}_{2}$ U(1): ${}^{1}_{1}+{}^{3}_{3}-{}^{2}_{2}-{}^{4}_{4},\ {}^{1}_{1}+{}^{2}_{2}+{}^{3}_{3}+{}^{4}_{4}-4\binom{5}{5}$
φ_6	SU(2)+U(1)+U(1)	SU(2): ${}^{1}_{3}\pm{}^{3}_{1}+{}^{2}_{4}\pm{}^{4}_{2},\ {}^{1}_{1}+{}^{2}_{2}-{}^{3}_{3}-{}^{4}_{4}$ U(1): ${}^{1}_{1}+{}^{3}_{3}-{}^{2}_{2}-{}^{4}_{4},\ {}^{1}_{1}+{}^{2}_{2}+{}^{3}_{3}+{}^{4}_{4}-4\binom{5}{5}$
φ_7	SU(2)+SU(2)+U(1)	SU(2): ${}^{1}_{4}\pm{}^{4}_{1},\ {}^{1}_{1}-{}^{4}_{4}$ SU(2): ${}^{2}_{3}\pm{}^{3}_{2},\ {}^{2}_{2}-{}^{3}_{3}$ U(1): ${}^{1}_{1}+{}^{2}_{2}+{}^{3}_{3}+{}^{4}_{4}-4\binom{5}{5}$
φ_8	SU(2)+SU(2)+U(1)	SU(2): ${}^{1}_{4}\pm{}^{4}_{1},\ {}^{1}_{1}-{}^{4}_{4}$ SU(2): ${}^{2}_{3}\pm{}^{3}_{2},\ {}^{2}_{2}-{}^{3}_{3}$ U(1): ${}^{1}_{1}+{}^{2}_{2}+{}^{3}_{3}+{}^{4}_{4}-4\binom{5}{5}$

[a] Non-maximal little algebra, maximal little group (Abud *et al* 1984a, b).
[b] Maximal little group.
Note. For convenience we have omitted a factor of i for the symmetric generators.

It is also possible to find vectors which map to the whole of the line between φ_2 and φ_3; these are also given in table 3, and shown in figure 2.

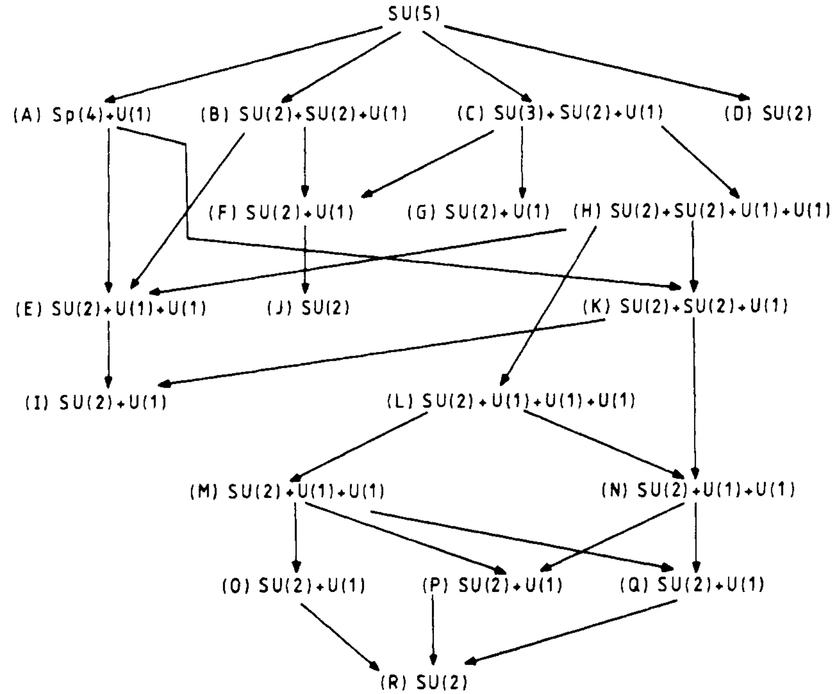
It is expected that not all of $\partial\Omega$ consists of straight lines, and so we turn to the problem of finding conditions for a curved section of the boundary to exist.

Consider first two vectors $\boldsymbol{\chi}_0$ and $\boldsymbol{\chi}_1 \in \mathbb{R}^{75}\backslash 0$, which map to distinct points $\boldsymbol{\psi}_0$, $\boldsymbol{\psi}_1$ on $\partial\Omega$. Assume that there are no cusps on the section of the boundary between $\boldsymbol{\psi}_0$ and $\boldsymbol{\psi}_1$, and that $\boldsymbol{\psi}_0$, $\boldsymbol{\psi}_1$ are sufficiently close together so that the radius of curvature of $\partial\Omega$ does not change sign between $\boldsymbol{\psi}_0$ and $\boldsymbol{\psi}_1$. Further consider a curve σ: $[0, 1] \rightarrow \mathbb{R}^{75}\backslash 0$ such that $\sigma(0) = \boldsymbol{\chi}_0$ and $\sigma(1) = \boldsymbol{\chi}_1$. Then (3.1) defines a curve in $\mathbb{R}^2$ from $\boldsymbol{\psi}_0$ to $\boldsymbol{\psi}_1$. The functional

$$\mathscr{L}(\sigma) = \int_0^1 L \, \mathrm{d}t \tag{4.10a}$$

where

$$L = \left[\left(\frac{\mathrm{d}X}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}Y}{\mathrm{d}t}\right)^2\right]^{1/2} \tag{4.10b}$$

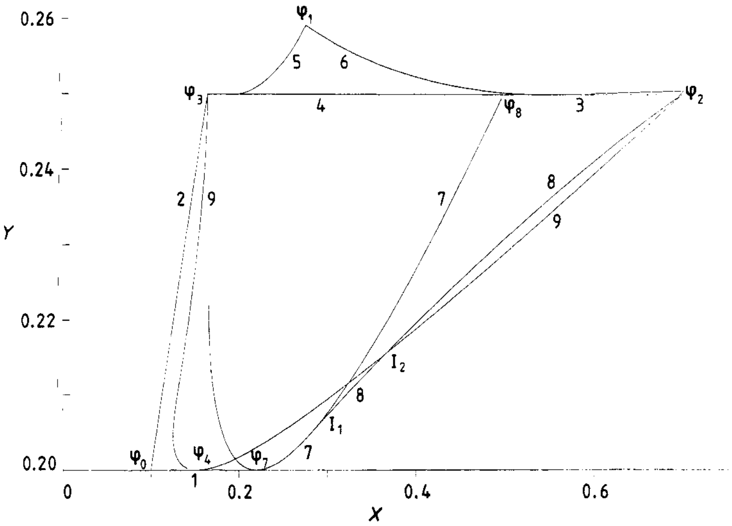


Letter in figure 1	Algebra	Embedding in fundamental representation of SU(5)	Number of invariants in representation $\{\bar{1}^2, 1^2\}$
A	Sp(4) + U(1)	$\langle 1\rangle_1+\langle 0\rangle_{-4}$	1
B	SU(2) + SU(2) + U(1)	$\{1\}\times\{1\}_1+\{0\}\times\{0\}_{-4}$	1
C	SU(3) + SU(2) + U(1)	$\{1\}\times\{0\}_2+\{0\}\times\{1\}_{-3}$	1
D	SU(2)	$\{4\}$	1
E	SU(2) + U(1) + U(1)	$\{1\}_{1,1}+\{1\}_{-1,1}+\{0\}_{0,-4}$	3
F	SU(2) + U(1)	$\{2\}_1+\{0\}_1+\{0\}_{-4}$	3
G	SU(2) + U(1)	$\{2\}_2+\{1\}_{-2}$	2
H	SU(2) + SU(2) + U(1) + U(1)	$\{1\}\times\{0\}_{1,1}+\{0\}\times\{1\}_{-1,1}+\{0\}_{0,-4}$	2
I	SU(2) + U(1)	$\{1\}_1+\{1\}_1+\{0\}_{-4}$	9
J	SU(2)	$\{2\}+\{0\}+\{0\}$	5
K	SU(2) + SU(2) + U(1)	$\{1\}\times\{0\}_1+\{0\}\times\{1\}_1+\{0\}_{-4}$	4
L	SU(2) + U(1) + U(1) + U(1)	$\{1\}_{1,1,0}+\{0\}_{-1,1,-1}+\{0\}_{-1,1,1}+\{0\}_{0,-4,0}$	3
M	SU(2) + U(1) + U(1)	$\{1\}_{1,1}+\{0\}_{-1,1}+\{0\}_{-1,1}+\{0\}_{0,-4}$	5
N	SU(2) + U(1) + U(1)	$\{1\}_{1,1}+\{0\}_{2,0}+\{0\}_{-4,4}+\{0\}_{2,2}$	5
O	SU(2) + U(1)	$\{1\}_3+\{0\}_{-2}+\{0\}_{-2}+\{0\}_{-2}$	9
P	SU(2) + U(1)	$\{1\}_1+\{0\}_1+\{0\}_1+\{0\}_{-4}$	7
Q	SU(2) + U(1)	$\{1\}_{-1}+\{0\}_4+\{0\}_4+\{0\}_{-6}$	9
R	SU(2)	$\{1\}+\{0\}+\{0\}+\{0\}$	15

Figure 1. Lattice of little algebras, excluding the U(1) + U(2) + ... algebras.

is the length of this curve. Thus if $\sigma(t)$ maps to a boundary it must be an extremum of $\mathscr{L}(\sigma)$. Writing

$$\frac{\mathrm{d}X}{\mathrm{d}t}=\frac{\partial X}{\partial\varphi_\alpha}\frac{\mathrm{d}\varphi_\alpha}{\mathrm{d}t}=X_\alpha\dot{\varphi}_\alpha \qquad \alpha=1,\ldots,75 \tag{4.11}$$



Curve/point in figure 2	Comment
$\varphi_1-\varphi_8$	See description in tables 1 and 2
1	See table 3
2	See table 3
3	See table 3
4	See table 3
5	Part of boundary generated by (4.18), see also table 5
6	Part of boundary generated by (4.17*a*), see also table 4
7	Part of boundary generated by (4.17*c*), see also table 4
8	Part of boundary generated by (4.17*b*), see also table 4
9	Part of boundary generated by (4.18), see also table 5
I_1	Intersection of curves 7 and 8, curve 7 ceases to be a boundary
I_2	Intersection of curves 8 and 9

Figure 2. The region Ω.

Table 3. Parametric form of straight lines in figure 2.

Parametric form of φ	Range of t	Line in figure 2	Generic little algebra	Generators
$-{}^{14}_{14}+{}^{15}_{15}+{}^{23}_{23}-{}^{25}_{25}-{}^{35}_{35}+{}^{45}_{45}+\sqrt{2}[{}^{23}_{14}+{}^{14}_{23}]$ $+t[{}^{12}_{34}+{}^{34}_{12}]+t[{}^{13}_{24}+{}^{24}_{13}]$	$0\leqslant t\leqslant 1$	1	U(1)+U(1)+U(1)	${}^1_1+{}^2_2+{}^3_3+{}^4_4-4({}^5_5)$ ${}^1_4+{}^4_1,\ {}^2_3+{}^3_2$
$-{}^{12}_{12}+{}^{13}_{13}+t^{14}_{14}-t^{15}_{15}-t^{23}_{23}+{}^{24}_{24}+t^{25}_{25}-{}^{34}_{34}$ $+t^{35}_{35}-t^{45}_{45}+\sqrt{(1-t^2)}[{}^{14}_{23}+{}^{23}_{14}]$	$0\leqslant t\leqslant 1$	2	U(1)+U(1)+U(1)	${}^1_1+{}^2_2+{}^3_3+{}^4_4-4({}^5_5)$ ${}^1_1-{}^4_4,\ {}^2_2-{}^3_3$
$2t^{14}_{14}+2t^{23}_{23}-t^{12}_{12}-t^{13}_{13}-t^{24}_{24}-t^{34}_{34}$ $+3[{}^{14}_{23}+{}^{23}_{14}]$	$0\leqslant t\leqslant 1$	3	SU(2)+SU(2)+U(1)	${}^1_1+{}^2_2+{}^3_3+{}^4_4-4({}^5_5)$ ${}^1_4\pm{}^4_1,\ {}^1_1-{}^4_4$ ${}^2_3\pm{}^3_2,\ {}^2_2-{}^3_3$
$t^{13}_{13}+t^{24}_{24}-t^{12}_{12}-t^{34}_{34}+{}^{14}_{23}+{}^{23}_{14}$	$0\leqslant t\leqslant 1$	4	U(1)+U(1)+U(1)	${}^1_1+{}^2_2+{}^3_3+{}^4_4-4({}^5_5)$ ${}^1_1-{}^4_4,\ {}^2_2-{}^3_3$

and treating this as a variational problem yields the Euler-Lagrange equations

$$L^{-1}\dot{L}[\dot{X}X_\alpha + \dot{Y}Y_\alpha] = [\ddot{X}X_\alpha + \ddot{Y}Y_\alpha] \tag{4.12}$$

which is of the form

$$m(t)X_\alpha + n(t)Y_\alpha = 0 \tag{4.13}$$

with $m(t)$ and $n(t)$ independent of α. Provided $m(t) \neq 0$ and $n(t) \neq 0$, this implies

either (i) $X_\alpha = Y_\alpha = 0$ (4.14a)

or (ii) $X_\alpha / Y_\alpha = -n(t)/m(t)$. (4.14$b$)

These conditions have been noted by Frautschi and Kim (1982). These equations are in general impossible to solve explicitly because of the large number of components involved. It is, however, possible to simplify (4.13) by noting that X and Y are both SU(5) invariant functions. Consequently (as pointed out by several authors (Michel and Radicati 1971, Abud and Sartori 1983, Jaric 1984)), at φ_α both X_α and Y_α are G_φ invariant vectors. Thus we may choose a subgroup $\mathrm{H} \subset \mathrm{G}$ and restrict σ to lie in Fix(H), the space of H invariant vectors, and look for solutions of (4.13) restricted to Fix(H), since all other components of (4.13) must vanish.

In general this problem will still be intractable unless H has only a small number of invariants. Fortunately, in this case, the curved part of $\partial\Omega$ does seem to correspond to such subgroups.

A systematic procedure for investigating all relevant subgroups H is to calculate the lattice of conjugacy classes of little groups of SU(5) for the representation $\{\bar{1}^2; 1^2\}$. This is a somewhat difficult task and a simpler method (which reduces to linear algebra) is to find the lattice of conjugacy classes of little algebras of SU(5) in this representation. To each algebra we may then associate a subgroup H of SU(5) by matrix exponentiation. These subgroups are the identity components of the various little groups of SU(5). Now if $\bar{\mathrm{H}}$ is a little group with identity component H it follows that H is a subgroup of $\bar{\mathrm{H}}$ and so Fix $\bar{\mathrm{H}} \subset$ Fix H. Hence by examining the spaces Fix H we include the spaces Fix $\bar{\mathrm{H}}$. Care must be taken, however, since some information has been lost. In particular important little groups may have small little algebras (see the note in the conclusion concerning φ_0).

Figure 1 displays the lattice of little algebras larger than $\mathrm{U}(1)^k$, together with the embedding of the subgroup H in the fundamental representation of SU(5) and the dimension of Fix (H). For an explanation of the notation used in this figure, and the method of calculation of dim (Fix (H)) see appendix 3.

We have examined the cases in figure 1 numerically by plotting random points in the XY plane generated by random points in each Fix (H). The results of this investigation are consistent with the conclusion that the curved part of $\partial\Omega$ is generated by the two spaces invariant under the algebras

$$\mathrm{SU(2)+SU(2)+U(1)} \qquad \{1\} \to \{1\} \times \{0\}_1 + \{0\} \times \{1\}_1 + \{0\} \times \{0\}_{-4} \tag{4.15}$$

see table 4 and

$$\mathrm{SU(2)+U(1)+U(1)} \qquad \{1\} \to \{1\}_{1,1} + \{1\}_{-1,1} + \{0\}_{0,-4} \tag{4.16}$$

see table 5.

Using (4.13) and (4.14) for the embedding (4.15) yields the explicit solutions (in the notation of table 4)

$$A^2 = \frac{69 - 225B^2 + (441 + 10\,710B^2 - 27\,135B^4)^{1/2}}{720} \qquad 0 \leqslant B^2 \leqslant \tfrac{1}{9} \tag{4.17a}$$

$$A^2=\frac{69-225B^2-(441+10\,710B^2-27\,135B^4)^{1/2}}{720} \qquad 0\leqslant B^2\leqslant\frac{29-\sqrt{193}}{108} \tag{4.17b}$$

$$A=0 \qquad 0\leqslant B\leqslant\tfrac{1}{3}. \tag{4.17c}$$

Table 4. SU(2) + SU(2) + U(1).

Embedding	$\{1\}\to\{1\}\times\{0\}_1+\{0\}\times\{1\}_1+\{0\}\times\{0\}_{-4}$
Branching	$\{\bar{1}^2;1^2\}\to4(\{0\}\times\{0\})_0+4(\{1\}\times\{1\})_0+2(\{0\}\times\{1\})_{\pm5}$ $+2(\{1\}\times\{0\})_{\pm5}+(\{2\}\times\{0\})_0+(\{0\}\times\{2\})_0$ $+(\{2\}\times\{2\})_0+(\{1\}\times\{2\})_{\pm5}+(\{2\}\times\{1\})_{\pm5}$
Generators of little group	${1\atop4}\pm{4\atop1}\;\;{1\atop1}-{4\atop4}$ ${2\atop3}\pm{3\atop2}\;\;{2\atop2}-{3\atop3}$ ${1\atop1}+{2\atop2}+{3\atop3}+{4\atop4}-4\left({5\atop5}\right)$
Invariant vectors (equivalent to φ_1, φ_2, φ_7)	$A\left(2\left({14\atop14}\right)+2\left({23\atop23}\right)-{12\atop12}-{13\atop13}-{24\atop24}-{34\atop34}\right)$ $B\left({14\atop14}-{15\atop15}-{23\atop23}+{25\atop25}+{35\atop35}-{45\atop45}\right)$ $C^{14}_{23}+\bar{C}^{23}_{14}$
Q	$8[6A^2+3B^2+\|C\|^2]$
K_2	$32[18A^4+3B^4+\|C\|^4+2[12A^2B^2+12A^2\|C\|^2+B^2\|C\|^2]$
K_3	$16[36A^4+8B^4+\|C\|^4+2[10A^2B^2+3A^2\|C\|^2+B^2\|C\|^2]$
Set $Q=8$ $\|C\|^2=1-6A^2-3B^2$	$X=\frac{1}{2}[-90A^4+6B^4+12A^2-4B^2-24A^2B^2+1]$ $Y=\frac{1}{4}[5B^4-2B^2+16A^2B^2+1]$

Table 5. SU(2) + U(1) + U(1).

Embedding	$\{1\}\to\{1\}_{1,1}+\{1\}_{-1,1}+\{0\}_{0,-4}$
Branching	$\{\bar{1}^2;1^2\}\to3\{0\}_{0,0}+2\{0\}_{\pm2,0}+\{0\}_{\pm4,0}$ $+2\{1\}_{\pm1,\pm5}+\{1\}_{\pm3,\pm5}+3\{2\}_{0,0}$ $+2\{2\}_{\pm2,0}+\{3\}_{\pm1,\pm5}+\{4\}_{0,0}$
Generators of little group	${1\atop4}+{2\atop3},\;{4\atop1}+{3\atop2},\;{1\atop1}-{4\atop4}+{2\atop2}-{3\atop3},\;{1\atop1}+{4\atop4}-{2\atop2}-{3\atop3},$ ${1\atop1}+{2\atop2}+{3\atop3}+{4\atop4}-4\left({5\atop5}\right)$
Invariant vectors (equivalent φ_1, φ_2, φ_3)	$P\left[{14\atop14}+{23\atop23}-{13\atop13}-{24\atop24}-{13\atop24}-{24\atop13}\right]$ $Q\left[{14\atop14}+{23\atop23}-{12\atop12}-{34\atop34}+{13\atop24}+{24\atop13}\right]$ $R\left[{14\atop14}-{23\atop23}-{15\atop15}-{45\atop45}+{25\atop25}+{35\atop35}\right]$
Q	$24[P^2+Q^2+R^2]$
K_2	$16[2Q^4+4R^4+(P+Q+R)^4+(P+Q-R)^4+2(P+(Q-R)^2)^2$ $+8P^2(Q^2-R)^2]$
K_3	$8[3P^2+3Q^2+2R^2+2PR+2QR]^2+8[3P^2+3Q^2-2R^2-2PR-2QR]^2$ $+64R^4$
Set $Q=24$ $R^2=1-P^2-Q^2$	$X=\frac{1}{6}[2P^4-8P^3+4P^2Q^2-4PQ^3+4PQ+1]$ $Y=\frac{1}{36}[P^4-8P^3Q+2P^2Q^2-8PQ^3+Q^4+8PQ+8]$

(4.16) does not yield an explicit solution, but the condition (in the notation of table 5)

$$Q^4 - Q^2(11P^2 - 1) + 2QP(2P^2 - 1) + P^2(2P^2 - 3) = 0. \tag{4.18}$$

These curves are shown in figure 2. It is clear that the conditions (4.13) are necessary, but not sufficient conditions for a curved boundary, and so some portions do not correspond to boundaries.

The reason for this is twofold. Firstly the equations (4.13) take into account only the first variation of $\mathscr{L}(\sigma)$. It is thus quite possible for the solutions to be of saddle-point type. This is in fact the case for the solution (4.17*c*). Secondly the degeneracy of the map (3.1) has not been taken into account. Thus two local solutions of (3.1) may project down onto two intersecting curves in $\mathbb{R}^2$. This appears to be the situation for the two curves (4.17*b*) and (4.18) which intersect at I_2 in figure 2.

It is not difficult, however, to verify by computer that (4.17) and (4.18) do indeed give the boundaries when the problem is restricted to the relevant Fix (H).

In view of the computational scale of investigating little groups with more than fifteen invariants this seems to exhaust the possibilities of this method. Thus the existence of a more complicated structure for $\partial\Omega$, particularly between φ_2 and φ_7, cannot be ruled out. However the final picture we obtain for $\partial\Omega$ in figure 3 shares several features in common with the results of Frautschi and Kim (1982) for the vector plus adjoint breaking of SU(5), and we believe that main features of $\partial\Omega$ have been exposed by our analysis.

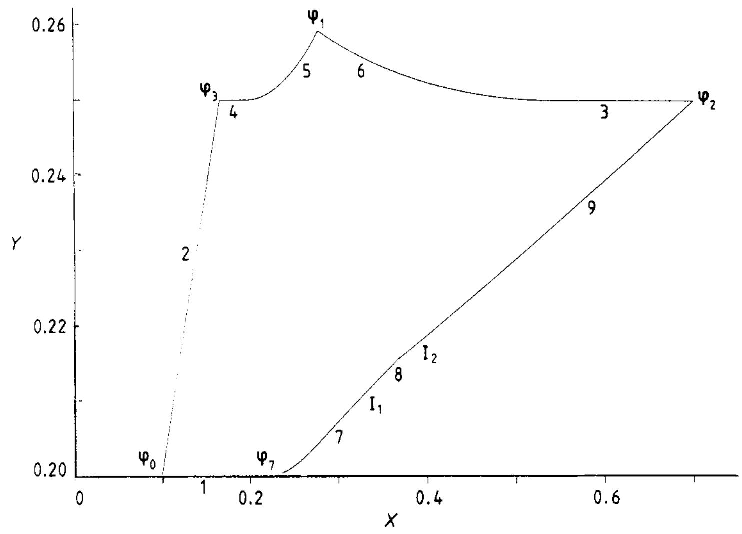


Figure 3. Boundary of Ω.

5. Conclusion

It has been shown that the most general Higgs potential of the **75** dimensional representation of $SU(5)\times Z_2$ is given by (2.7). The task of minimising the potential reduces to finding the boundary of a closed, bounded, connected subset, Ω, of $\mathbb{R}^2$ defined by the map (3.1).

A combination of methods leads to the results shown in figure 3 for this boundary, with the reservations made at the end of § 4.

To determine the possible symmetry breaking patterns we use the fact, as noted in § 3, that the minimisation of $V(\varphi)$ is equivalent to minimising the function $\lambda_1+\lambda_2 X+\lambda_3 Y$ on Ω. This leads immediately, using figure 4, to the results of table 6.

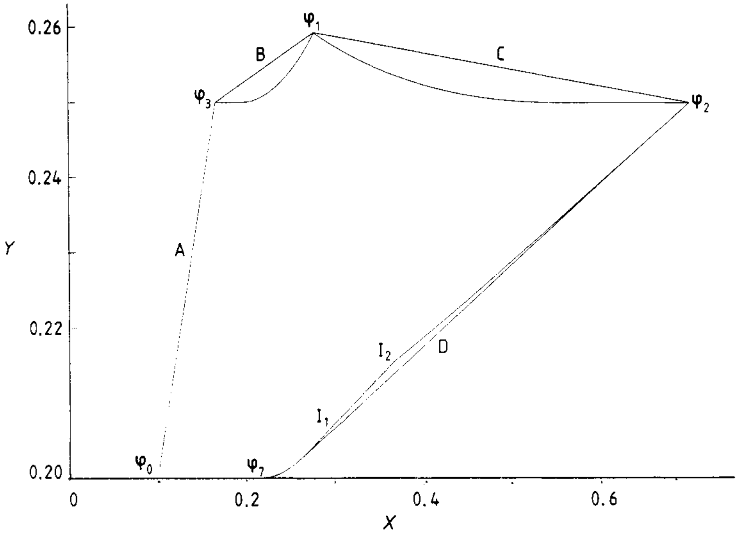


Figure 4. Equations of straight lines enclosing Ω. A, $Y=\frac{3}{4}X+\frac{1}{8}$; B, $Y=\frac{1}{12}X+\frac{17}{72}$; C, $Y=-\frac{5}{228}X+\frac{121}{456}$; D, $Y=\frac{1}{64}(15-\sqrt{65})X+\frac{1}{640}(55+7\sqrt{65})$.

Table 6. Symmetry breaking patterns.

Range of parameters	Position in figure 2	Sub-algebra	Embedding
$\lambda_2>0,\ \lambda_3<0,\ 0\leqslant-\lambda_2/\lambda_3<\frac{3}{4}$ and $\lambda_2<0,\ \lambda_3<0,\ 0\leqslant\lambda_2/\lambda_3<\frac{5}{228}$	φ_1	SU(3)+SU(2)+U(1)	$(\{1\}\times\{0\})_2+(\{0\}\times\{1\})_{-3}$
$\lambda_2>0,\ \lambda_3<0,\ \frac{1}{12}<-\lambda_2/\lambda_3<\frac{3}{4}$	φ_3	SU(2)+SU(2)+U(1) ($\simeq$SO(4)+U(1))	$(\{0\}\times\{0\})_{-4}+(\{1\}\times\{1\})_1$
$\lambda_2<0,\ \lambda_3<0,\ \frac{5}{228}<\lambda_2/\lambda_3\leqslant\infty$ and $\lambda_2<0,\ \lambda_3>0,\ C<-\lambda_2/\lambda_3\leqslant\infty$	φ_2	Sp(4)+U(1)	$\langle 1\rangle_1+\langle 0\rangle_{-4}$
$\lambda_2,\ \lambda_3>0$ and $\lambda_2>0,\ \lambda_3<0,\ \frac{3}{4}<-\lambda_2/\lambda_3\leqslant\infty$	φ_0	U(1)+U(1)+U(1) +U(1)	Maximal torus
$\lambda_2<0,\ \lambda_3>0,\ 0<-\lambda_2/\lambda_3<C$	Curve 7	SU(2)+SU(2)+U(1)	$(\{1\}\times\{0\})_1+(\{0\}\times\{1\})_1$ $+(\{0\}\times\{0\})_{-4}$

$C=\frac{1}{64}(-15+\sqrt{65})$

There are several interesting points to raise:

(i) The existence of an SU(3)+SU(2)+U(1) minimum has been noted before (Abud *et al* 1984a, Hubsch *et al* 1984). Our results confirm this finding and show that for an open set of the parameters this is an *absolute minimum.* Thus from the point of view of model building the **75** appears to be a viable alternative to the usual adjoint.

(ii) The existence of the U(1)+U(1)+U(1)+U(1) absolute minimum is interesting because it arises in an algebraic rather than a group theoretic way. Indeed an approach based on calculating the lattice of figure 1 would probably not have found this minimum.

This case is also an example of the crucial distinction between a little group and a little algebra (the Lie algebra of a little group). It is clear that U(1)+U(1)+U(1)+U(1) is not a maximal little algebra. The little group of φ_0, G_{φ_0}, is however maximal. This can be proven by noting that φ_0 is invariant under the Z_5 symmetry generated by the permutation

$$\pi=\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 2 & 4 & 1 & 5 & 3\end{pmatrix}.$$

It is not difficult to show that the only vector invariant under both U(1)×U(1)×U(1)×U(1) and this Z_5 symmetry is (up to a multiplicative constant) φ_0. Thus since any G_{φ_0} invariant vector must be invariant under both these groups it follows that G_{φ_0} has only one invariant and so is maximal.

(iii) The SU(2)+SU(2)+U(1) absolute minima (curve 7 in figure 2) correspond to a non-maximal symmetry breaking pattern and so violate Michel's conjecture (Michel 1979). This is similar to the example discovered by Abud *et al* (1984b), which was not, however, shown to be an absolute minimum. Once again care must be taken to distinguish between algebras and groups. It is clear from figure 1 that the algebra SU(2)+SU(2)+U(1) is not maximal. To show that these minima also have non-maximal little groups we note, using (4.21*c*) and table 4, that the vectors which map to curve 7 in figure 2 have the generic form

$$\varphi=B\left({}^{14}_{14}-{}^{15}_{15}-{}^{23}_{23}+{}^{25}_{25}+{}^{35}_{35}-{}^{45}_{45}\right)+C\,{}^{14}_{23}+\bar{C}\,{}^{23}_{14}\qquad B\neq 0,\ C\neq 0. \tag{5.1}$$

Computing the matrix $J_a^b=\varphi^{ac}_{de}\varphi^{fg}_{bc}\varphi^{de}_{fg}=\varphi^{ac}_{de}A^{de}_{bc}$ yields

$$J=4B|C|^2\begin{pmatrix}1 &&&& \\ & -1 &&& \\ && -1 && \\ &&& 1 & \\ &&&& 0\end{pmatrix} \tag{5.2}$$

Since J_a^b is constructed from φ it commutes with any representation matrix D_a^b of G_φ. Thus G_φ is conjugate to a group of block diagonal form

$$\begin{pmatrix}G_1 && \\ & G_2 & \\ && Z\end{pmatrix} \tag{5.3}$$

where G_1, G_2 are 2×2 unitary matrices and Z is a unimodular complex number. Requiring that φ be invariant under G_φ implies that $\det G_1=\det G_2$.

By inspection these matrices (after a suitable conjugation) also leave φ_1 invariant and it follows that G_φ is not a maximal little group.

If G_φ is a subgroup of $U(5)\times Z_2$, the full linear symmetry group of the potential, then Z is arbitrary, whereas if G_φ is to be a subgroup of SU(5) then $Z=(\det G_1)^{-2}$. In both cases the above argument holds and G_φ is a non-maximal little group.

Acknowledgments

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Appendix 1

Littlewood (1944a, b) in a highly original approach to the invariant theory of the classical groups made extensive use of Schur functions (S functions) and the operation of plethysm which he had earlier introduced as a new type of multiplication (Littlewood 1950a p 206, 1950b p 289).

Each S function $\{\mu\}$, labelled by a partition $(\mu)=(\mu_1, \mu_2 \ldots \mu_p)$ with $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_p > 0$, $p \leqslant N$, corresponds to an irreducible covariant tensor representation of the unitary group $U(N)$. This correspondence arises in the following manner. If $D(g)$ is the representation of the element g of $U(N)$, then the character of g in this representation is simply $\mathrm{Tr}\, D(g)$. Since this expression is invariant under transformations $g \to UgU^{-1}$, where U is a unitary matrix, g may be diagonalised. Thus $\mathrm{Tr}\, D(g)$ depends only on the eigenvalues of g. It follows that the character is a polynomial function of N variables with certain symmetry properties. It is this function which is the S function, $\{\mu\}$, the symmetry properties being described by the partition (μ). Since a representation is completely specified by its character there is a one to one correspondence between S functions and representations of $U(N)$. As a consequence of this definition the partition (μ) also specifies the symmetry of the indices of the tensor representation.

To each irreducible representation $\{\mu\}$ there corresponds a complex conjugate or contragredient representation. This is an irreducible contraviariant tensor representation whose character $\{\bar{\mu}\}$ is the complex conjugate of $\{\mu\}$. The partition (μ) again specifies the symmetry of the tensor indices. The characters of mixed irreducible tensor representations may be conveniently denoted by $\{\bar{\nu};\mu\}$ (King 1970), where the partitions μ and ν specify the symmetry properties of the covariant and contravariant tensor indices and the semicolon indicates tracelessness. With this notation the character of the representation **75** of U(5) realised by the fields ϕ^{ab}_{cd} is given by $\{\bar{1}^2; 1^2\}$.

Quite generally the number of algebraically independent invariants of degree n in the components of φ is given by the number of times the identity representation appears in the plethysm $\{\nu_G\}\otimes\{n\}$ where $\{n\}$ is a one part partition and φ realises the representation $\{\nu_G\}$. This plethysm is nothing other than the nth-fold symmetrised power of $\{\nu_G\}$. Thus in the case of the **75** of U(5) it is necessary to consider $\{\bar{1}^2; 1^2\}\otimes\{n\}$ with $n=2$–4 to find the number of invariants appearing in the most general Higgs potential.

It might be thought that since the group under consideration is SU(5) rather than U(5), mixed tensors may be dispensed with altogether. In principle this is correct and using ε_{abcde} to lower contravariant indices it is easy to see (King 1970) that the character

$\{\bar{1}^2; 1^2\}$ of SU(5) is equivalent to $\{2^21\}$. Unfortunately current tabulations of plethysms do not extend to the plethysm $\{2^21\}\otimes\{4\}$.

To circumvent this difficulty it seems worthwhile spelling out the rules for dealing with S functions and their plethysms, extending the usual techniques to cover the case of mixed tensor representations.

Firstly there are three different products of S functions:

(*i*) *Outer products*

$$\{\mu\}\cdot\{\nu\}=\sum_{\rho}\Gamma_{\mu\nu\rho}\{\rho\}. \tag{A1.1}$$

This product is associated with the tensor of Kronecker product of representations of U(N) and the corresponding multiplication of their characters. Under this operation S functions form a commutative, associative algebra. The product may be evaluated using the Littlewood-Richardson rule (Littlewood 1950a p 94, Hammermesh 1962 p 250). Various tabulations of this product exist (Itzykson and Nauenberg 1966, Wybourne 1970).

(*ii*) *Inner products*

$$\{\mu\}\circ\{\nu\}=\sum_{\rho}g_{\mu\nu\rho}\{\rho\}. \tag{A1.2}$$

This operation owes its existence to the duality between the unitary and symmetric groups. Necessarily (μ), (ν) and (ρ) are partitions of the same number n and the product is nothing other than the corresponding Kronecker product for irreducible representations of S_n, the symmetric group on n elements. Various techniques exist for evaluating such products (Robinson 1961 p 64) and tables of inner products have been published (Itzykson and Nauenberg 1966, Wybourne 1970).

(*iii*) *Plethysms*

$$\{\mu\}\otimes\{\nu\}=\sum_{\rho}p_{\mu\nu\rho}\{\rho\}. \tag{A1.3}$$

This operation may be thought of as that of forming the nth-fold symmetrised power of $\{\mu\}$ if (ν) is a partition of n. Under this operation S functions do not form an algebra since it is not distributive over addition on the left. The symbol $\otimes$ should not be confused with that for tensor product. It is retained out of deference to the work of Littlewood.

(Tables of plethysms may be found in Butler and Wybourne (1971).) Plethysms obey the following rules

$$\{\mu\}\otimes(\{\nu\}\cdot\{\rho\})=(\{\mu\}\otimes\{\nu\})\cdot(\{\mu\}\otimes\{\rho\}) \tag{A1.4a}$$

$$\{\mu\}\otimes(\{\nu\}\pm\{\rho\})=(\{\mu\}\otimes\{\nu\})\pm(\{\mu\}\otimes\{\rho\}) \tag{A1.4b}$$

$$(\{\mu\}\otimes\{\nu\})\otimes\{\rho\}=\{\mu\}\otimes(\{\nu\}\otimes\{\rho\}) \tag{A1.4c}$$

$$(\{\mu\}+\{\nu\})\otimes\{\rho\}=\sum_{\sigma\tau}\Gamma_{\sigma\tau\rho}(\{\mu\}\otimes\{\sigma\})\cdot(\{\nu\}\otimes\{\tau\}) \tag{A1.4d}$$

$$(\{\mu\}-\{\nu\})\otimes\{\rho\}=\sum_{\sigma\tau}(-1)^t\Gamma_{\sigma\tau\rho}(\{\mu\}\otimes\{\sigma\})\cdot(\{\nu\}\otimes\{\tau'\}) \qquad (\tau)\vdash t \tag{A1.4e}$$

$$(\{\mu\}\cdot\{\nu\})\otimes\{\rho\}=\sum_{\sigma\tau}g_{\sigma\tau\rho}(\{\mu\}\otimes\{\sigma\})\cdot(\{\nu\}\otimes\{\tau\}) \tag{A1.4f}$$

where (τ') is the partition conjugate to (τ), and $(\tau)\vdash t$ means that (τ) is a partition of

t. The coefficients $g_{\sigma\tau\rho}$ are totally symmetric under all permutations of σ, τ and ρ, whilst $g_{\sigma\tau'\rho'} = g_{\sigma\tau\rho}$. The following special cases are of interest here

$$\{0\}\otimes\{\mu\} = \begin{cases} \{0\} & \text{if } \{\mu\} = \{m\} \\ \varnothing & \text{if } \{\mu\} \neq \{m\} \end{cases} \tag{A1.5a}$$

$$\{\mu\}\otimes\{0\} = \{0\}$$

$$\{1\}\otimes\{\mu\} = \{\mu\}\otimes\{1\} = \{\mu\} \tag{A1.5b}$$

$$(\{\mu\}+\{\nu\})\otimes\{m\} = \sum_{t=0}^{m} (\{\mu\}\otimes\{m-t\})\cdot(\{\nu\}\otimes\{t\}) \tag{A1.6a}$$

$$(\{\mu\}-\{\nu\})\otimes\{m\} = \sum_{t=0}^{m} (-1)^t(\{\mu\}\otimes\{m-t\})\cdot(\{\nu\}\otimes\{1^t\}) \tag{A1.6b}$$

$$(\{\mu\}\cdot\{\nu\})\otimes\{m\} = \sum_{\sigma\vdash m} \{\mu\}\otimes\{\sigma\}\cdot\{\nu\}\otimes\{\sigma\} \tag{A1.6c}$$

$$(\{\mu\}\cdot\{\nu\})\otimes\{1^m\} = \sum_{\sigma\vdash m} \{\mu\}\otimes\{\sigma\}\cdot\{\nu\}\otimes\{\sigma'\} \tag{A1.6d}$$

where (m) is a partition of one part.

Extending these rules to include mixed tensor representations is straightforward through the use of the further rules

$$\{\bar{\mu}\}\cdot\{\bar{\nu}\} = \overline{\{\mu\}\cdot\{\nu\}} \tag{A1.7a}$$

$$\{\bar{\mu}\}\otimes\{\nu\} = \overline{\{\mu\}\otimes\{\nu\}} \tag{A1.7b}$$

$$\{\bar{\mu}\}\cdot\{\nu\} = \sum_{\tau} \{\overline{\mu/\tau};\, \nu/\tau\} \tag{A1.7c}$$

$$\{\bar{\mu};\, \nu\} = \sum_{\tau} (-1)^t\{\overline{\mu/\tau}\}\cdot\{\nu/\tau'\} \qquad (\tau)\vdash t \tag{A1.7d}$$

where

$$\{\mu/\tau\} = \sum_{\rho} \Gamma_{\tau\rho\mu}\{\rho\}. \tag{A1.8}$$

It should be noted that (A1.7d) corrects a crucial misprint in an earlier paper (King 1975). It follows from (A1.7d) that

$$\{\bar{1}^2;\, 1^2\} = \{\bar{1}^2\}\cdot\{1^2\} - \{\bar{1}\}\cdot\{1\}. \tag{A1.9}$$

Combining this with the other rules yields the following example of the plethysm of mixed S functions

$$\begin{aligned}\{\bar{1}^2;\, 1^2\}\otimes\{2\} &= \{\bar{1}^2\cdot 1^2 - \bar{1}\cdot 1\}\otimes 2 \\ &= (\bar{1}^2\cdot 1^2)\otimes 2 - \bar{1}^2\cdot 1^2\cdot\bar{1}\cdot 1 + (\bar{1}\cdot 1)\otimes 1^2 \\ &= \bar{1}^2\otimes 2\cdot 1^2\otimes 2 + \bar{1}^2\otimes 1^2\cdot 1^2\otimes 1^2 - \bar{1}^2\cdot\bar{1}\cdot 1^2\cdot 1 \\ &\quad + \bar{1}\otimes 2\cdot 1\otimes 1^2 + \bar{1}\otimes 1^2\cdot 1\otimes 2 \\ &= (\bar{2}^2+\bar{1}^4)\cdot(2^2+1^4) + \overline{21}^2\cdot 21^2 - (\overline{21}+\bar{1}^3)\cdot(21+1^3) + \bar{2}\cdot 1^2 + \bar{1}^2\cdot 2.\end{aligned} \tag{A1.10a}$$

Using (A1.7c) yields after the cancellation of a large number of terms

$$\begin{aligned}\{\bar{1}^2;1^2\}\otimes\{2\}=&\{\bar{2}^2;2^2\}+\{\bar{2}^2;1^4\}+\{\bar{1}^4;2^2\}+\{\bar{1}^4;1^4\}\\&+\{\overline{21}^2;21^2\}+\{\overline{21};21\}+\{\overline{21};1^3\}+\{\bar{1}^3;21\}\\&+\{\bar{1}^3;1^3\}+\{\bar{2};2\}+\{\bar{1}^2;1^2\}+\{\bar{1}^2;1^2\}+\{\bar{1};1\}+\{0\}.\end{aligned}\tag{A1.10b}$$

It is not difficult to see that in fact the rules (A1.1)-(A1.8) may be used to calculate any plethysm of mixed S functions.

From the point of view of invariant theory (A1.10b) leads to the trivial conclusion that the representation $\{\bar{1}^2; 1^2\}$ of U(N) contains a single second degree invariant, independent of N. This result is unaltered in the case of SU(N), and in the case of SU(5) is simply Q as given in (2.3) and (2.4).

In determining the number of invariants it is only necessary to find the number of times the identity representation, $\{0\}$, occurs in the plethysm. The amount of work involved is considerably reduced by noting that a product $\{\bar{\mu}\}\cdot\{\nu\}$ contains $\{0\}$ once and only once if and only if $\{\mu\}=\{\nu\}$. Making use of this for N sufficiently large yields for both U(N) and SU(N)

$$\{\bar{1}^2;1^2\}\otimes\{2\}=1\{0\}+\dots\tag{A1.11a}$$

$$\{\bar{1}^2;1^2\}\otimes\{3\}=2\{0\}+\dots\tag{A1.11b}$$

$$\{\bar{1}^2;1^2\}\otimes\{4\}=6\{0\}+\dots.\tag{A1.11c}$$

This confirms that (2.2) is indeed the most general U(N) or SU(N) invariant potential. For small N these results are not necessarily true. This is because of the existence of modification rules (King 1971) giving rise to various equivalence relations between representations of U(N) labelled by $\{\bar{\nu};\mu\}$. If the number of parts of the partitions (μ) and (ν) are given by respectively p and q then $\{\bar{\nu};\mu\}$ is standard if $p+q\leqslant N$. If $p+q>N$ then $\{\bar{\nu};\mu\}$ is either zero or $\pm\{\bar{\tau};\sigma\}$ for some standard character $\{\bar{\tau};\sigma\}$. For example $\{\bar{1}^2;1^2\}$ is standard for $N\geqslant 4$, but zero for $N=3$ and $-\{\bar{1};1\}$ for $N=2$.

Rather than applying modification rules to the final result such as (A1.10b) for a plethysm, it is much simpler to note that for U(N) or SU(N) $\{\mu\}=\varnothing$ (we use $\varnothing$ to make the crucial distinction between zero and the S function $\{0\}$) if the number of parts of the partition (μ) exceeds N, and to use this result at an intermediate stage of the calculation. In the case under consideration, provided that $\{\bar{1}^2;1^2\}$ is standard, and correspondingly $N\geqslant 4$, (A1.10a) implies that

$$\{\bar{1}^2;1^2\}\otimes 2=1\{0\}+\dots\qquad\text{for } N\geqslant 4.\tag{A1.12}$$

However

$$\begin{aligned}\{\bar{1}^2;1^2\}\otimes\{3\}=&\bar{3}^2\cdot 3^2+\overline{2^2}1^2\cdot 2^21^2+\bar{1}^6\cdot 1^6+\overline{321}\cdot 321+\overline{2^21}^2\cdot 2^21^2+\overline{21}^4\cdot 21^4\\&+\bar{2}^3\cdot 2^3+\overline{31}^3\cdot 31^3-\overline{32}\cdot 32-\overline{2^2}1\cdot 2^21-\overline{21}^3\cdot 21^3-\bar{1}^5\cdot 1^5-\overline{31}^2\cdot 31^2\\&-\overline{2^2}1\cdot 2^21-\overline{21}^3\cdot 21^3+\overline{21}^2\cdot 21^2+\overline{21}^2\cdot 21^2-\overline{21}\cdot 21+\dots.\end{aligned}\tag{A1.13}$$

Applying the modification rules then yields

$$\begin{aligned}\{\bar{1}^2;1^2\}\otimes\{3\}&=1\{0\}+\dots\qquad \mathrm{N}=4,5 &&\text{(A1.14a)}\\&=2\{0\}+\dots\qquad N\geqslant 6. &&\text{(A1.14b)}\end{aligned}$$

Similarly

$$\{\bar{1}^2; 1^2\} \otimes \{4\} = 6\{0\} + \ldots \qquad N \geqslant 8 \tag{A1.15a}$$

$$5\{0\} + \ldots \qquad N = 6, 7 \tag{A1.15b}$$

$$3\{0\} + \ldots \qquad N = 5 \tag{A1.15c}$$

$$2\{0\} + \ldots \qquad N = 4. \tag{A1.15d}$$

The particular case of these results for $N = 5$ imply that for SU(5) the potential $V(\varphi)$ can be written in terms of one quadratic, two cubic and three quartic invariants. Correspondingly there must exist identities of the number and type given in (2.5). Their derivation is discussed in appendix 2. The existence of these identities, together with the elimination of the cubic term by means of a discrete symmetry, then proves that (2.7) is the most general form of the required $\mathrm{SU}(5) \times Z_2$ invariant potential.

Appendix 2

Having established in appendix 1 the existence of relationships of the type and number of (2.5) it is necessary to derive their form explicitly. The origin of these identities can be traced to the vanishing of certain expressions due to an antisymmetrisation over more than five indices which take only the values 1 to 5.

Thus, using [. . .] to denote antisymmetrisation, the expansion of

$$\delta^{a}_{[g} \delta^{b}_{h} \varphi^{cd}_{cd} \varphi^{ef}_{ef]} = 0 \tag{A2.1}$$

yields, after some algebraic manipulation, the useful result:

$$\delta^{a}_{[g} \delta^{b}_{h]} Q - 4 \delta^{[a}_{[g} B^{b]}_{h]} + 4 A^{ab}_{gh} - 8 D^{ab}_{[gh]} = 0. \tag{A2.2}$$

Note that this equation only holds if the indices range from 1 to 5 since it is necessary to use $\delta^a_a = 5$. From (A2.2) can be derived all but one of the required identities (2.5). Firstly contraction with φ^{gh}_{ab} gives

$$A^{gh}_{ab} \varphi^{ab}_{gh} - 4 D^{gh}_{ab} \varphi^{ab}_{gh} = 0 \tag{A2.3}$$

that is

$$C = 4C'. \tag{A2.4}$$

Contracting with A^{gh}_{ab} yields

$$Q^2 - 8 B^b_h B^h_b + 2 A^{ab}_{gh} A^{gh}_{ab} - 8 D^{ab}_{gh} A^{gh}_{ab} = 0 \tag{A2.5}$$

that is

$$K_1 - 8K_3 + 2K_2 + 8K_4 = 0. \tag{A2.6}$$

Finally contracting with D^{gh}_{ab} gives

$$-Q^2 + 8K_3 - 4K_4 + 8K_5 - 8K_6 = 0. \tag{A2.7}$$

To find the last relation we may expand

$$\varphi^{\;xa}_{[\,ya} \varphi^{zb}_{tb} \varphi^{cd}_{cd\,]} = 0 \tag{A2.8}$$

which yields

$$\varphi^{a[x}_{yt} B^{z]}_{a} - 2 \varphi^{a[\;x}_{y\;\;c} D^{z]c}_{[ta]} - 2 \varphi^{a[x}_{td} D^{z]d}_{ay} - 2 D^{xz}_{cd} \varphi^{cd}_{yt} - \varphi^{a[x}_{b[y} A^{z]b}_{t]a} = 0 \tag{A2.9}$$

and contracting with φ^{yt}_{xz} then gives

$$-K_3+3K_4-2K_5+4K_6=0. \tag{A2.10}$$

The equations (2.5) follow immediately and are in agreement with Abud *et al* (1984b). This accounts for all the required identities and from the arguments of appendix 1 there can exist no others, although this does not exclude the possibility of further tensorial identities, provided they do not imply additional relationships between the second, third and fourth degree invariants.

It should be pointed out that (A2.2) is really an identity between covariants transforming like $\{\bar{1}^2; 1^2\}$. This technique of using obvious identities such as (A2.1) and (A2.8) is quite general and has been used, for example, to calculate well known identities satisfied by the Riemann curvature tensor (De Witt 1963) which has the symmetry $\{\bar{1}^2; 1^2\}$.

Appendix 3

The notation $\left(\begin{smallmatrix} i & j \\ k & l \end{smallmatrix}\right)$ is used in the tables to denote the basis vector $e^{[i}\otimes e^{j]}\otimes e_{[k}\otimes e_{l]}$ of $V^*\wedge V^*\otimes V\wedge V$ where $\{e_i\}$ $i=1,\ldots,5$ span a five-dimensional fundamental representation space V of SU(5). Thus $\varphi=\Sigma_{k,l,m,n}\,\varphi^{mn}_{kl}\left(\begin{smallmatrix} k & l \\ m & n \end{smallmatrix}\right)$ is a vector in the 75-dimensional representation of SU(5), provided φ^{mn}_{kl} satisfies $\varphi^{mn}_{ml}=0$ and $\varphi^{mn}_{kl}=(\varphi^{kl}_{mn})^*$.

The generators of SU(5) are represented by linear combinations of the matrices $\left(\begin{smallmatrix} i \\ j \end{smallmatrix}\right)$, where $\left(\begin{smallmatrix} i \\ j \end{smallmatrix}\right)^n_m=\delta^i_m\delta^n_j$. The generators are anti-Hermitian, but for convenience we have omitted the factor of i in the symmetric generators.

The action of $\left(\begin{smallmatrix} i \\ j \end{smallmatrix}\right)$ on $\left(\begin{smallmatrix} k & l \\ m & n \end{smallmatrix}\right)$ is simply

$$\begin{pmatrix} i \\ j \end{pmatrix}\cdot\begin{pmatrix} k & l \\ m & n \end{pmatrix}=-\delta^i_m\begin{pmatrix} k & l \\ j & n \end{pmatrix}-\delta^i_n\begin{pmatrix} k & l \\ m & j \end{pmatrix}+\delta^k_j\begin{pmatrix} i & l \\ m & n \end{pmatrix}+\delta^l_j\begin{pmatrix} k & i \\ m & n \end{pmatrix}. \tag{A3.1}$$

We also have the condition that T $(=\Sigma_{ij}\,T^i_j\left(\begin{smallmatrix} j \\ i \end{smallmatrix}\right))$ is a generator of G_φ if and only if

$$T\cdot\varphi=0. \tag{A3.2}$$

Thus if φ is given, then (A3.1) and the condition (A3.2) may be used to find the little algebra of φ. Alternatively given any sub-algebra $\mathscr{H}$, we may find the most general φ invariant under the group H generated by $\mathscr{H}$ by imposing the condition (A3.2) for each generator of $\mathscr{H}$.

The lattice of little algebras $\mathscr{H}$ for the representation $\{\bar{1}^2; 1^2\}$ of SU(5) are displayed in figure 1. The precise embeddings of $\mathscr{H}$ in SU(5) are indicated by giving in each case the representation of H obtained by restricting to this subgroup of SU(5) the defining five-dimensional representation $\{1\}$ of SU(5). This restriction takes the form

$$\mathrm{SU}(5)\supset\mathrm{H} \qquad \{1\}\to\sum_{\mu_\mathrm{H}} m_{\mu_\mathrm{H}}\mu_\mathrm{H} \tag{A3.3}$$

where μ_H denotes an irreducible representation of H. The notation in figure 1 is such that $\{1\}$ and $\langle 1\rangle$ denote the defining n-dimensional representations of SU(n) and Sp(n) respectively whilst $\{0\}$ and $\langle 0\rangle$ signify the trivial one-dimensional representations of these groups. This notation coincides with that of Littlewood (1950a) and Wybourne (1970). In addition subscripts $1, -4, 2, \ldots$ are used to denote corresponding representations $\{1\}$, $\{\bar{4}\}$, $\{2\}\ldots$ of U(1) wherever appropriate. It should be pointed out that groups H of the form $\mathrm{U}(1)\times\mathrm{U}(1)\times\ldots\times\mathrm{U}(1)$ have been omitted from the lattice of figure 1(a), there being far too many of these to display, all arising from further branching of the SU(2) representations.

Alongside each representation of H shown in figure 1(b) is given the dimension of the corresponding H invariant subspace, Fix (H), of the 75-dimensional space of the representation $\{\bar{1}^2; 1^2\}$ of SU (5). This dimension is most readily calculated by making use of (A1.9) and the theorem (see Wybourne 1970), that the branching of the representation $\{\bar{\rho}; \sigma\}$ of SU (5) to H is given by

$$\left(\sum_{\mu_H} m_{\mu_H}\mu_H\right)\otimes\{\bar{\rho}; \sigma\}. \tag{A3.4}$$

It follows, using the algebra of plethysms, and the fact that $\bar{\lambda}_H \cdot \rho_H$ contains the trivial one-dimensional representation of H once and only once if and only if $\lambda_H = \rho_H$, that

$$\dim(\mathrm{Fix}\ H) = \sum_{\nu_H} n_{\nu_H}^2 - \sum_{\mu_H} m_{\mu_H}^2 \tag{A3.5}$$

where the coefficients m_{μ_H} occur in the branching (A3.3) and the coefficients n_{ν_H} occur in the reduction

$$\mathrm{SU}(5) \supset \mathrm{H} \qquad \{1^2\} \to \sum_{\nu_H} n_{\nu_H}\nu_H. \tag{A3.6}$$

The branching coefficients may be found directly (using (A3.4)) or more simply by using well known general results (King 1975, 1982), or tabulated results (McKay and Patera 1981, Slansky 1981).

Note added in proof. Since completing this work we have discovered the paper by Hübsch *et al* (1985), which also discusses this problem, in particular the relations (2.5) were given in this paper using a somewhat different method of derivation.

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